5. Permutations and Combinations

- **Factorial notation:** The notation $n!$ represents the product of the first n natural numbers, i.e.,

$n! = n \times (n-1) \times (n-2) \times \ldots\ldots \times 5 \times 4 \times 3 \times 2 \times 1$
$0! = 1$

Example:

$$\frac{12!}{8!} = \frac{12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1}$$

$$= 12\times11\times10\times9$$

$$= 11880$$

- **Fundamental Principle of Counting:** If an event occurs in m different ways, following which another event occurs in n different ways, then the total number of occurrence of the events in the given order is $m \times n$. This is called the fundamental principle of counting.

Example:
Find the number of 5-letter words, with or without meaning, which can be formed out of the letters of the word MATHS, where the repetition of digits is not allowed.

Solution:

There are as many words as there are ways of filling 5 vacant places □□□□□ by the 5 letters.
The first place can be filled with any of the 5 letters in 5 different ways, following which the second place can be filled with any of the remaining 4 letters in 4 different ways, following which the third place can be filled in 3 different ways, following which the fourth place can be in 2 different ways, following which the fifth place can be filled in 1 way.
Thus, the number of ways in which the 5 places can be filled, by the multiplication principle, is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Note: If repetition of letters had been allowed, then the required number of words would be $5 \times 5 \times 5 \times 5 \times 5 = 3125$.

- **Permutation when all objects are distinct:** A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

The number of permutations of n different things taken r at a time, when

- repetition is not allowed, is ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
- repetition is allowed, is n^r, where $0 \le r \le n$.

Example 1: Twenty five students are participating in a competition. In how many ways, can the first three prizes be won in such a way that a prize cannot be shared by more than one student?

Solution: The total number of ways in which first three prizes can be won is the number of arrangements of 25 different things taken 3 at a time.

So, required number of ways $= {}^{25}P_3$

$= \frac{25!}{(25-3)!}$

$= \frac{25!}{22!}$

$= \frac{25 \times 24 \times 23 \times 22!}{22!}$

$= 25 \times 24 \times 23 = 13800$

Example 2: Find the total number of four digit numbers that can be formed by using the digits 0, 2, 5, and 6?

Solution: A four digit number has four places i.e., units, tens, hundreds and thousands. Units, tens and hundreds place can be filled with either 0, 2, 5, or 6 where as thousands place can be filled with 2, 5 or 6 only.

Number of ways to fill the units place = 4
Number of ways to fill the tens place = 4
Number of ways to fill the hundreds place = 4
Number of ways to fill the thousands place = 3 ways.
∴ Total number of four digit numbers = 4 × 4 × 4 × 3 = 192

- **Concept of permutations when all objects are not distinct**
    - The number of permutations of n objects, when p objects are of the same kind and the rest are all different, is $\frac{n!}{p!}$.
        - In general, the number of permutations of n objects, when p_1 objects are of one kind, p_2 are of the second kind , … , p_k are of the k^{th} kind and the rest, if any, are of different kinds, is $\frac{n!}{p_1!\, p_2! \ldots\ldots p_k!}$.

Example: Find the number of permutations of the letters of the word ARRANGEMENT.

Solution: Here, there are 11 objects (letters) of which there are 2A's, 2R's, 2N's, 2E's and the rest are all different.
∴ Required number of arrangements

$= \frac{11!}{2!2!2!2!}$

$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 2}{2 \times 2 \times 2 \times 2}$

$= 2494800$

Circular Permutation

When the arrangements of n different things are around a circle, the arrangements are circular permutations. Circular arrangements are considered different when the relative order of the things is changed.

- The number of circular permutations of n different things is $(n-1)!$.
- The number of circular permutations of n different things taken all at a time is n-1!2 if clockwise and anti-clockwise orders are not different.
- The number of circular permutations of n different things taken r at a time is Prnr or Crnr-1! .
- The number of circular permutations of n different things taken r at a time is Prn2r or Crnr-1!2 if clockwise and anti-clockwise orders are not different.

- The number of circular permutations of n things, of which r are alike, is n-1!r! .

- **Combinations:** The number of combinations of n different things taken r at a time is denoted by ${}^{n}C_{r}$, which is given by

$$ {}^{n}C_{r} = \frac{n!}{r!(n-r)!}, \text{ where } 0 \le r \le n. $$

In particular, ${}^{n}C_{0} = {}^{n}C_{n} = 1$

Example 1: A box contains 8 red bulbs and 5 blue bulbs. Determine the number of ways in which 4 red and 2 blue bulbs can be selected.
Solution:
It is given that a box contains 8 red bulbs and 5 blue bulbs.
Now, 4 red bulbs can be selected from 8 red bulbs in ${}^{8}C_{4}$ number of ways, and 2 blue bulbs can be selected from 4 blue bulbs in ${}^{4}C_{2}$ number of ways.
Hence, 4 red bulbs and 2 blue bulbs can be selected from a box containing 8 red bulbs and 4 blue bulbs in ${}^{8}C_{4} \times {}^{4}C_{2}$ number of ways.
Now,

$$ {}^{8}C_{4} \times {}^{4}C_{2} $$
$$ = \frac{8!}{4! \times (8-4)!} \times \frac{4!}{2!(4-2)!} $$
$$ = \frac{8!}{4! \times 4!} \times \frac{4!}{2!2!} $$
$$ = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} $$
$$ = 70 \times 6 $$
$$ = 420 $$

Thus, the number of ways of selecting the bulbs is 420.

- $${}^{n}C_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = {}^{n}C_{r}$$

In other words, selecting r objects out of n objects is the same as rejecting $(n-r)$ objects.

- ${}^{n}C_{a} = {}^{n}C_{b} \Rightarrow a = b$ or $a = n$, i.e., $n = a + b$

Example 2: If ${}^{19}C_{3r} = {}^{19}C_{2r+4}$, then find the value of r.
Solution:
${}^{19}C_{3r} = {}^{19}C_{2r+4}$
$\Rightarrow 3r + (2r + 4) = 19$ or $3r = 2r + 4$
$\Rightarrow 3r + (2r + 4) = 19$ $\Rightarrow$ r = 4
$\Rightarrow 5r + 4 = 19$
$\Rightarrow 5r = 19 - 4 = 15$
$\Rightarrow r = 3$

$\therefore$ The value of r is either 3 or 4.

- ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

Example 3: If ${}^{n}C_{r-1} + {}^{n}C_{r} + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} = {}^{n+a}C_{r+(a-1)}$, then find the value of a.

Solution:

${}^{n}C_{r-1} + {}^{n}C_{r} + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$

$= {}^{n+1}C_{r} + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$ $\quad (\because {}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r})$

$= {}^{n+2}C_{r+1} + {}^{n+2}C_{r+2}$

$= {}^{n+3}C_{r+2}$

$\therefore {}^{n+3}C_{r+2} = {}^{n+a}C_{r+(a-1)}$

$\Rightarrow a = 3$